Model predictive control for uncertain nonlinear systems subject to chance constraints

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Abstract— This paper presents a model predictive control scheme for a class of stochastic nonlinear systems subject to chance constraints. The applied control is composed of a nominal control action which is based on the solution of a tightened deterministic optimal control problem and an ancillary control law. The ancillary control law fights the uncertainties in between sampling times. Chance constraints satisfaction as well as convergence in a probabilistic sense of the closed loop system are discussed. The overall approach is only slightly more computational expensive than the corresponding nominal, deterministic model predictive controller. The approach is illustrated by a numerical example.

I. INTRODUCTION

Model predictive control (MPC) has received great attention over the past decades. The success of MPC is mainly due to its capability to explicitly handle state and input constraints. In "standard" deterministic MPC [1, 2] a finite horizon open-loop optimization problem based on the current system states is solved. Only the first part of the obtained input trajectory is applied to the system, and the optimal control problem is solved again using the updated state, leading to an update of the input.

However, many systems are uncertain or with respect to disturbances. Robust MPC schemes have been proposed to guarantee stability and constraints satisfaction, which mainly consider bounded disturbances or uncertainties. The assumption of bounded disturbances or uncertainties does not hold or lead to conservative behavior. One way to tackle such systems is to consider the stochastic disturbances and to use Monte Carlo based approaches [3,4]. Other MPC schemes, also used here, handle stochastic uncertainties by predicting the behavior of a suitable nominal system, and replacing the chance constraints by tightened deterministic constraints. In case that the disturbances are Gaussian and the system is linear the resulting problem can be relaxed to a standard deterministically constrained MPC problem [5]. In [6–9], stochastic expansions of so called tube based MPC for linear systems are proposed. Often, Cantelli's inequality is used to replace the chance constraints by deterministic constraints [6,7]. This, however, is limited to constraints, of which each is a linear function of one state variable.

Multivariate Chebyshev's inequality is used, for example, in [9], allowing the constraints to be linear combinations of states and inputs. Chance constraints are handled directly using the concept of probabilistic invariance in [10, 11], there confidence ellipsoids in the state space, or polytopic sets that are known to contain the vector of uncertain parameters with a given probability are used. Stochastic MPC for nonlinear systems has received only little attention, c.f., [12]. Under the assumption that the optimal stochastic cost function is continuous, stability of model predictive control for continuous-time stochastic systems is studied in [13]. Model predictive control of nonlinear systems subject to probabilistic parametric uncertainties exploiting polynomial chaos expansions has been investigated in many works [14, 15], establishing strict stability results for polynomial chaos based methods is, however, challenging.

In this paper, a stochastic model predictive control scheme for a class of nonlinear systems with chance constraints is proposed. Multivariate Chebyshev's inequality is used to reformulate stochastic chance constraints into deterministic constraints. The control law is composed of an open-loop control part and a closed-loop, permanently applied ancillary control law. The open loop control input is obtained by solving repeatedly a finite horizon optimization problem with tightened constraints. The approach guarantees recursive feasibility, and has almost the same online computational burden as the nominal MPC for deterministic systems.

In Section II the problem setup and necessary preliminary results are stated. The control scheme, including the chance constraints, and the ancillary control law, is introduced in Section III. Chance constraints satisfaction and convergence results are presented in Section IV. In Section V, a simulation example demonstrates the features of the proposed scheme. Section VI concludes the paper with a brief summary.

Basic Nomenclature and Definitions:

For an random vector s, E[s] and $C_{ov}[s]$ denote the expectation and covariance matrix of s, respectively. Given two events A and B, $P_r(A|B)$ denotes the conditional probability of A given B. For a symmetric matrix $X \in \mathbb{R}^{n \times n}$, $X \succ 0(X \succeq 0)$ denotes that X is a positive (semi-) definite matrix, and $X \prec 0(X \preceq 0)$ denotes that X is a negative (semi-) definite matrix. Furthermore, $||s||_M := \sqrt{s^T M s}$ for $M = M^T \succeq 0$. The Pontryagin difference of $A \in \mathbb{R}^n$ and $B \in \mathbb{R}^n$ is defined as [16] $A \ominus B = \{x \in \mathbb{R}^n | x + y \in A, \forall y \in B\}$, while the Minkowski sum of A and B is given by $A \oplus B = \{\varpi \in \mathbb{R}^n \mid \exists x \in A, y \in B : \varpi = x + y\}$. The

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multiplication of a set $\mathcal{B} \subset \mathbb{R}^n$ by a matrix $A \in \mathbb{R}^{m \times n}$ is given by [17] $A\mathcal{B} = \{c \in \mathbb{R}^m \mid \exists b \in \mathcal{B}, c = Ab\}.$

II. PROBLEM SETUP & PRELIMINARIES

We consider a continuous-time nonlinear system with respect to stochastic disturbances

$$\dot{x} = Ax + g(x) + Bu + B_w w, \tag{1}$$

where the term $g(\cdot)$ is assumed to be differentiable, and g(0) = 0. The disturbances w(t) with E[w] = 0 and $C_{ov}[w_i(s), w_i(t)] \leq \delta(t-s)\alpha^2$ are independent, identically distributed noise, where $\delta(\cdot)$ is a Dirac delta function, and $\alpha > 0$ is a given constant.

The control input and system state have to satisfy chance constraints of the form

$$P_r \{x(t) \in \mathcal{X}\} \ge p,$$

$$P_r \{u(t) \in \mathcal{U}\} \ge p, \quad \forall t \ge 0,$$
(2)

with $p \in (0, 1)$, where \mathcal{X} and \mathcal{U} are compact sets. Perfect state information is assumed to be available at all times.

We denote

$$\dot{\bar{x}} = A\bar{x} + g(\bar{x}) + B\bar{u},\tag{3}$$

as the nominal system.

The applied control signal consists of a nominal, openloop, input \bar{u} and a locally time-varying state feedback K(t)v, i.e., $u = \bar{u} + K(t)v$, where $v := x - \bar{x}$ denotes the error between the real system (1) and the nominal system (3). The error dynamics are given by

$$v := Av + g(x) - g(\bar{x}) + BK(t)v + B_w w.$$
(4)

The objective of this paper is to develop an MPC scheme for the systems (1) that allows to satisfy the chance constraints and is computational attractive. The basic idea is to calculate the open-loop input at discrete sampling times, to predict the nominal dynamics (3) with the nominal control input \bar{u} , and to tight the disturbances in between samplings by the linear control law such that the probabilistic constraints are satisfied.

III. MODEL PREDICTIVE CONTROL

In this section we outline the MPC scheme consisting of an online calculated open-loop input based on the repeatedly solution of a nominal MPC problem at the sampling times, and a local control law calculated offline which is used to keep the uncertain trajectory close to the nominal trajectory.

A. Ancillary control law

First, note that since g(h) is differentiable, there always exists a $\tilde{x} := (1 - c)x + c\bar{x}$ for some $c \in [0, 1]$ such that Eq.(4) can be rewritten as

$$\dot{v} = Av + g(x) - g(\bar{x}) + BK(t)v + B_w w$$

$$= Av + \frac{dg(h)}{dh}|_{h=\bar{x}}v + BK(t)v + B_w w \qquad (5)$$

$$= A(t)v + BK(t)v + B_w w,$$

where $A(t) \triangleq A + \frac{dg(h)}{dh}|_{h=\tilde{x}}$.

Assumption 1: There exists a locally control law K(t) such that A(t)+BK(t) is asymptotically stable for all $t \ge 0$. Suppose that v(0) = 0, then the mean of the error v(t) is given by

$$E[v(t)] = E\left[\int_0^t e^{A_{cl}(\tau)(t-\tau)} B_w w(\tau) d\tau\right]$$
$$= \int_0^t e^{A_{cl}(\tau)(t-\tau)} B_w E[w(\tau)] d\tau = 0$$

with $A_{cl}(t) \triangleq A(t) + BK(t)$.

Furthermore, the covariance of v(t) is given by

$$E \left[v(t)v(t)^{T} \right]$$

= $E \left[\int_{0}^{t} \int_{0}^{t} e^{A_{cl}(\tau_{1})(t-\tau_{1})} B_{w}w(\tau_{1}) w(\tau_{2})^{T} B_{w}^{T} e^{A_{cl}(\tau_{2})^{T}(t-\tau_{2})} d\tau_{1} d\tau_{2} \right]$
= $\int_{0}^{t} \int_{0}^{t} e^{A_{cl}(\tau_{1})(t-\tau_{1})} B_{w} E \left[w(\tau_{1})w(\tau_{2})^{T} \right] B_{w}^{T} e^{A_{cl}(\tau_{2})^{T}(t-\tau_{2})} d\tau_{1} d\tau_{2}$
 $\leq \alpha^{2} \int_{0}^{t} e^{A_{cl}(\tau)(t-\tau)} B_{w} B_{w}^{T} e^{A_{cl}(\tau)^{T}(t-\tau)} d\tau.$

Thus, with $\Gamma(t) := \int_0^t e^{A_{cl}(\tau)(t-\tau)} B_w B_w^T e^{A_{cl}(\tau)^T(t-\tau)} d\tau$, we have that

$$E\left[v(t)v(t)^T\right] \preceq \alpha^2 \Gamma(t).$$

 $\Gamma(t)$ is a time-varying controllability gramian of (A_{cl}, B_w) [18, 19], which is the solution of

$$\frac{d\Gamma(t)}{dt} = B_w B_w^T + A_{cl}(t)\Gamma(t) + \Gamma(t)A_{cl}^T(t), \Gamma(0) = 0.$$
(6)

Since $e^{A_{cl}(\tau)(t-\tau)}B_w B_w^T e^{A_{cl}(\tau)^T(t-\tau)} \succ 0$, $\Gamma(t)$ is monotonically increasing with time t. Suppose that $\lim_{t\to\infty} \Gamma(t)$ exists. Without loss of generality, denote $\lim_{t\to\infty} \Gamma(t) = \Gamma$. Then, $\Gamma \geq \Gamma(t)$ for all $t \geq 0$.

We propose to use linear differential inclusion (LDI) to obtain static linear control gain K and Γ . Suppose that for a static linear control law Kv and for each $\tilde{x} \in \mathcal{X}$ there exists a linear differential inclusion Ω such that

$$\begin{bmatrix} A_{cl}(t) & B_w \end{bmatrix} \in \Omega, \forall \tilde{x} \in \mathcal{X} \text{ and } t.$$
(7)

Then every trajectory of the error system (5) is also a trajectory of the LDI defined by Ω .

Suppose that the linear differential inclusion Ω is a polytope described by a list of its vertices,

$$\Omega := Co\left\{ \begin{bmatrix} A_1 & B_w \end{bmatrix}, \cdots, \begin{bmatrix} A_N & B_w \end{bmatrix} \right\}$$

where N is the number of vertex. Then, (6) can be reduced to an algebraic matrix equation, if there exist timeinvariant matrices Γ and K such that (6) is satisfied for all $[A_i \ B_w] \in \Omega$, $i \in [1, N]$. Moreover, the corresponding algebraic matrix equation can be solved by linear matrix inequalities (LMIs).

B. Deterministic reformulation of the chance constraints

Satisfication of the chance constraints in a deterministic MPC problem requires to reformulate them in a deterministic way. For shortness of presentation, we denote the covariance of v as Σ , i.e., $\Sigma := C_{ov}[v]$, and $H = \alpha^2 \Gamma$. Furthermore, for a fixed $\epsilon > 0$, we define the two sets

$$\mathcal{D}_{\Sigma} = \left\{ v \in \mathbb{R}^{n_x} \mid v^T \Sigma^{-1} v \le \epsilon \right\} \\ \mathcal{D}_H = \left\{ v \in \mathbb{R}^{n_x} \mid v^T H^{-1} v \le \epsilon \right\}.$$

Then, if E[v] = 0 and $E[vv^T] \preceq \alpha^2 \Gamma$, we have that $\Sigma \preceq H$ and $\mathcal{D}_{\Sigma} \subseteq \mathcal{D}_H$. Furthermore, due to Multivariate Chebyshev's inequality [20,21], $P_r\{v \in \mathcal{D}_{\Sigma}\} \ge 1 - \frac{n_x}{\epsilon}$. Thus, $P_r\{v \in \mathcal{D}_H\} \ge 1 - \frac{n_x}{\epsilon}$, and the state of the error system satisfies $P_r\{v^T H^{-1}v \le \epsilon\} \ge 1 - \frac{n_x}{\epsilon}$. Choosing $p = 1 - \frac{n_x}{\epsilon}$, we know that the system states are in \mathcal{D}_H with probability p:

$$P_r\left\{v(t) \in \mathcal{D}_H\right\} \ge p, \forall t \ge 0.$$
(8)

Choosing the sets \mathcal{X}_0 and \mathcal{U}_0 as follows

$$\begin{aligned}
\mathcal{X}_0 &= \mathcal{X} \odot \mathcal{D}_H \\
\mathcal{U}_0 &= \mathcal{U} \odot K \mathcal{D}_H,
\end{aligned}$$
(9)

then the constraints (2) are satisfied with probability p if $\bar{x} \in \mathcal{X}_0$ and $\bar{u} \in \mathcal{U}_0$.

C. Sampled-data open-loop control input

Similar as in [22–24], the opne-loop control input is obtained by the solution of an optimal control problem subject to the nominal dynamics (3). To achieve satisfaction of the probability constraints, tightened, deterministic constraints $\bar{x} \in \mathcal{X}_0$ and $\bar{u} \in \mathcal{U}_0$ are used. As the cost function we use

$$J(\bar{x}(t_k), \bar{u}(\cdot; \bar{x}(t_k))) := S\left(\bar{x}(t_k + T_p, \bar{x}(t_k))\right) + \int_{t_k}^{t_k + T_p} \|\bar{x}\left(\tau; \bar{x}(t_k)\right)\|_Q^2 + \|\bar{u}\left(\tau; \bar{x}(t_k)\right)\|_R^2 d\tau,$$
(10)

where T_p is the prediction horizon, $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are positive definite weighting matrices.

We use a terminal set $\mathcal{X}_f := \{\bar{x} \in \mathbb{R}^{n_x} \mid S(\bar{x}) \leq \alpha\}$ with $\alpha > 0$ and a terminal penalty $S(\bar{x})$ to guarantee stability and recursive constraint satisfication. To this end, we assume that:

Assumption 2: (Terminal region, terminal penalty) There exists a control law $\pi(\bar{x})$ such that [2,25]:

B0. $\mathcal{X}_f \subseteq \mathcal{X}_0$, B1. $\pi(\bar{x}) \in \mathcal{U}_0$ for all $\bar{x} \in \mathcal{X}_f$, B2. $S(\bar{x})$ satisfies $\alpha_3(\|\bar{x}\|) \leq S(\bar{x}) \leq \alpha_4(\|\bar{x}\|)$, and $\frac{\partial S(\bar{x})}{\partial \bar{x}} (A\bar{x} + g(\bar{x}) + B\bar{u}) + l(\bar{x}, \pi(\bar{x})) \leq 0$ (11)

for all
$$\bar{x} \in \mathcal{X}_f$$
, where $\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ are class \mathcal{K}_{∞} functions, and $l(x, u) := \|x\|_Q^2 + \|u\|_R^2$.

We will use two optimization problems to derive feasible open-loop input for satisfication of the chance constraints. Problem 1: (nominal case)

Pro

$$\begin{array}{l} \underset{\bar{x}(t_k),\bar{u}(\cdot;\bar{x}(t_k))}{\mini} J(\bar{x}(t_k),\bar{u}(\cdot;\bar{x}(t_k))) \\ \text{subject to} \\ \dot{\bar{x}} = A\bar{x} + g(\bar{x}) + B\bar{u}, \\ x(t_k) - \bar{x}(t_k) \in \mathcal{D}_H, \\ \bar{x}(\tau;\bar{x}(t_k)) \in \mathcal{X}_0, \quad \tau \in [t_k, t_k + T_p), \\ \bar{u}(\tau;\bar{x}(t_k)) \in \mathcal{U}_0, \quad \tau \in [t_k, t_k + T_p), \\ \bar{x}(t_k + T_p; \bar{x}(t_k)) \in \mathcal{X}_f. \end{array}$$

Compared to Problem 1, $x(t_k) - \bar{x}(t_k) \in \mathcal{D}_H$ is added and $\bar{x}(t_k)$ is optimized in Problem 2. Furthermore, in Problem 2 directly the updated system state $x(t_k)$ enters, which captures the present disturbances. In the following, the $\bar{u}^*(\tau, \bar{x}(t_k))$ denotes the optimal input, and $\bar{x}^*(\cdot; \bar{x}(t_k))$ the predicted trajectory of (3) starting from $\bar{x}(t_k)$ driven by the optimal input $\bar{u}^*(\cdot; \bar{x}(t_k))$, where $\tau \in [t_k, t_k + T_p]$.

For implementation, either Problem 1 or Problem 2 is solved. For simplicity, we consider an equidistant sampling time δ , $t_k = \delta \cdot k$. The nominal control during the sampling interval, the times t_k when the open-loop problem is solved, is given by

$$\bar{u}(\tau) = \bar{u}^*\left(\tau; \bar{x}(t_k)\right), \quad \tau \in [t_k, t_k + \delta).$$
(12)

The applied control input to the system is given

$$u(\tau) := \bar{u}(\tau) + K(x(\tau) - x^*(\tau; \bar{x}(t_k))), \tau \in [t_k, t_k + \delta).$$

Note the continuous feedback term, given by the ancillary control law $K(\cdot)$, is introduced.

It is known that sampled-data MPC stabilizes the nominal system if the terminal conditions B0-B2 are satisfied [1,2]. Furthermore, the optimal value function J^* satisfies

$$J^{*}(\bar{x}(t+\delta,\bar{x}(t)),\cdot) - J^{*}(\bar{x}(t),\cdot)$$

$$\leq -\int_{0}^{\delta} \|\bar{x}(t+\tau,\bar{x}(t))\|_{Q}^{2} + \|\bar{u}(t+\tau,\bar{x}(t))\|_{R}^{2})d\tau,(13)$$

which will be used to prove the convergence of the real systems under control.

The proposed MPC scheme is formally described by:

- Algorithm 1: (Nonlinear MPC with ancillary control law) Step 0. At initial time t_0 , let $\bar{x}(t_0) = x(t_0)$ where $x(t_0)$ is the current state. Solve Problem 1 to obtain the nominal control input $\bar{u}(\tau)$ and the applied control input $u(\tau), \tau \in [t_0, t_0 + \delta]$. Precede to Step 2.
 - Step 1. At time t_k , nominal state $\bar{x}(t_k)$ and real state $x(t_k)$:

(i) If $x(t_k) - \bar{x}(t_k) \in \mathcal{D}_H$, solve Problem 2 to obtain

ū(τ), *x̄*(t_k), and calculate u(τ), τ ∈ [t_k, t_k + δ].
(ii) If x(t_k) − *x̄*(t_k) ∉ D_H, solve Problem 1 to obtain *ū*(τ), and calculate u(τ), τ ∈ [t_k, t_k + δ].
Step 2. Apply u(τ) for τ ∈ [t_k, t_k + δ].

Step 2. Apply a(r) for $r \in [t_k, t_k + 0]$. Step 3. Obtain $x(t_{k+1})$ and calculate $\bar{x}(t_{k+1})$ for $t_{k+1} := t_k + \delta$, set k := k + 1, precede to Step 1.

IV. CHANCE CONSTRAINTS SATISFACTION & CONVERGENCE

We discuss the properties of the closed-loop system under control, i.e., chance constraints satisfaction and convergence of the system dynamics in a probabilistic sense.

A. Chance constraints satisfaction

We emphasize that $x(t) - \bar{x}(t) \in \mathcal{D}_H$ with probability pfor all t if $x(t_0) = \bar{x}(t_0)$ for some t_0 . That is, $x(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$ with probability p for all t since $\bar{x}(t) \in \mathcal{X}_0$ and $\bar{u}(t) \in \mathcal{U}_0$ for all t. For simplicity, we use $t_{n+1} := t_n + \delta$.

Three cases are considered to prove chance constraints satisfaction.

Event A: $v(t_n) = 0$,

For $t \in [t_n, t_n + T_p]$, there exists $\bar{u}(t, \bar{x}(t_n))$ such that

- $\bar{x}(t, \bar{x}(t_n)) \in \mathcal{X}_0, \ \bar{u}(t, \bar{x}(t_n)) \in \mathcal{U}_0$
- $\bar{x}(t_n + T_p, \bar{x}(t_n)) \in \mathcal{X}_f.$
- **Event B:** $v(t_{n+1}) \in \mathcal{D}_H$,

For $t \in [t_{n+1}, t_n + T_p]$,

•
$$\bar{x}(t, \bar{x}(t_{n+1})) \in \mathcal{X}_0, \ \bar{u}(t, \bar{x}(t_{n+1})) \in \mathcal{U}_0$$

• $\bar{x}(t_n + T_p, \bar{x}(t_{n+1})) \in \mathcal{X}_f,$

and for $t \in [t_n + T_p, t_{n+1} + T_p]$,

- a nominal trajectory starts from $\bar{x}(t_n + T_p, \bar{x}(t_{n+1}))$ under the control law $\pi(\cdot)$.
- **Event C:** $v(t_n) = 0$, and $v(t_{n+1}) \in \mathcal{D}_H$,
- For $t \in [t_n, t_{n+1} + T_p]$, there exists $\bar{u}(t, \bar{x}(t_n))$ such that
 - $\bar{x}(t,\bar{x}(t_n)) \in \mathcal{X}_0, \ \bar{u}(t,\bar{x}(t_n)) \in \mathcal{U}_0$
 - $\bar{x}(t_{n+1}+T_p,\bar{x}(t_n)) \in \mathcal{X}_f.$

and for $t \in [t_n + T_p, t_{n+1} + T_p]$,

• a nominal trajectory starts from $\bar{x}(t_n + T_p, \bar{x}(t_{n+1}))$ under the control law $\pi(\cdot)$.

Note that, for Event A, the actual system states $x(t, \bar{x}(t_n))$ lie in the set \mathcal{D}_H centered the predicted trajectory $\bar{x}(t, \bar{x}(t_n))$ with probability p which is driven by the control $\bar{u}(t, x(t_n))$, $t \in [t_n, t_n + T_p]$, since $x(t, \bar{x}(t_n)) = \bar{x}(t, \bar{x}(t_n)) + v(t)$ and $v(t) \in \mathcal{D}_H$ with probability p.

Theorem 1: Suppose that Problem 1 is feasible at time t_0 . Then, the chance constraints (2) are satisfied with probability p for all $t \ge t_0$.

Proof: Two scenarios are considered: $v(t_{n+1}) \in \mathcal{D}_H$ and $v(t_{n+1}) \notin \mathcal{D}_H$ for all $t \in [t_{n+1}, t_0 + T_p]$.

(1) Suppose that $v(t_{n+1}) \in \mathcal{D}_H$, choose $\bar{x}(t, \bar{x}(t_{n+1})) := \bar{x}(t, \bar{x}(t_n))$ and $\bar{u}(t, \bar{x}(t_{n+1})) := \bar{u}(t, \bar{x}(t_n))$ for all $t \in [t_{n+1}, t_n + T_p]$, then the first three conditions of Event B are satisfied. That is, the Second and Third condition of Event B can be chosen from the segment trajectory of Event A, see the blue line of Fig.1. Once $\bar{x}(t_n + T_p, \bar{x}(t_n))$ enters into the terminal set \mathcal{X}_f , it will never leave the terminal set \mathcal{X}_f , cf.



Fig. 1. Sketch idea of recursive feasibility

Assumption 2. Thus there exists $\bar{u}(t_n + \tau, \bar{x}(t_n)) \in \mathcal{U}_0$ such that $\bar{x}(t_n + \tau, \bar{x}(t_n)) \in \mathcal{X}_0$ for all $\tau \in [t_n + T_p, t_{n+1} + T_p]$. Due to the discussion above, the occurrence of Event A implies the occurrence of Event B, i.e., $P_r(B|A) = 1$.

Since Problem 1 is feasible at time t_n , and $P_r\{v(t) \in D_H\} \ge p$ for all $t \ge 0$, $P_r(A) \ge p$. Furthermore, $P_r(C) = P_r(A \cap B) = P_r(B|A)P_r(A) \ge p$ in terms of $P_r(A) \ge p$ and $P_r(B|A) = 1$.

Moreover, for Event A, $x(t, \bar{x}(t_n)) \in \mathcal{X}$, $u(t, \bar{x}(t_n)) \in \mathcal{U}$ with probability p for all $t \in [t_n, t_n + T_p]$ in terms of $\mathcal{X}_0 = \mathcal{X} \odot \mathcal{D}_H$ and $\mathcal{U}_0 = \mathcal{U} \odot K \mathcal{D}_H$. The same holds for Event C, i.e., both the actual state constraints and the actual input constraints are satisfied in Event C.

(2) Suppose that $v(t_{\delta}) \notin \mathcal{D}_H$, then Problem 1 is solved. Due to Theorem 1 in [9], chance constraints are satisfied with probability p.

In terms of induction, chance constraints (2) are satisfied with probability p for all $t \ge t_0$ if Problem 1 is feasible at time t_0 .

Note that chance constraints are satisfied for all $t \ge 0$ since the disturbances w(t) can be regarded as a realization of w(t) at t = 0.

B. Convergence

Define a function

$$V(x) = J(\bar{x}^*, \bar{u}^*(\cdot, \bar{x}^*)), \tag{14}$$

which is the optimal cost function of the related problem. Next properties of the optimal cost function are derived which will be used to prove that the real system state converges and stays in the set \mathcal{D}_H in a probabilistic sense as time goes to infinity.

Lemma 1: Suppose that Problem 1 is feasible at time t_0 with $\bar{x}(t_0) = x(t_0)$, then,

- (i) $0 \leq V(x) < +\infty$,
- (ii) V(x) = 0, for all $x \in \mathcal{D}_H$, and

$$V(x(t+\delta)) - V(x(t)) \leq -\int_{t}^{t+\delta} \|\bar{x}^{*}(\tau, \bar{x}(t))\|_{Q}^{2} + \|\bar{u}^{*}(\tau, \bar{x}(t))\|_{R}^{2} d\tau$$
(15)

Proof: (i) follows directly from the definition of $V(\cdot)$.

(iii)

(ii) Let x(t) be an arbitrary point in \mathcal{D}_H . Since $x(t) \in 0 \oplus \mathcal{D}_H$, it follows that $\bar{x}^*(t) = 0$ and $\bar{u}^*(\tau, 0) \equiv 0$, for all $\tau \geq t$, consist of a feasible solution to Problem 2. Hence $V(x) \leq J(0, \bar{u}^*(\tau, 0)) = 0$, which establishes the result.

(iii) If $x(t + \delta) - \bar{x}^* (t + \delta, \bar{x}(t)) \in \mathcal{D}_H$, $\bar{x}^* (t + \delta, \bar{x}(t))$, $\bar{u}^* (\tau, \bar{x}(t))$ for $\tau \in [t + \delta, t + T_p]$ and the control law $\pi(\cdot)$ for $\tau \in [t + T_p, t + \delta + T_p]$ consist of a feasible solution to Problem 2 at the time instant $t + \delta$.

If $x(t + \delta) - \bar{x}^* (t + \delta, \bar{x}(t)) \notin \mathcal{D}_H$, $\bar{u}^* (\tau, \bar{x}(t))$ for $\tau \in [t+\delta, t+T_p]$ and the control law $\pi(\cdot)$ for $\tau \in [t+T_p, t+\delta+T_p]$ consist of a feasible solution to Problem 1 at the time instant $t + \delta$.

In terms of Eq. (13), $J(\bar{x}^*(t+\delta,\bar{x}(t)),\cdot) - J(\bar{x}^*(t),\cdot) \leq -\int_0^\delta \|\bar{x}(t+\tau,\bar{x}(t))\|_Q^2 + \|\bar{u}(t+\tau,\bar{x}(t))\|_R^2 d\tau$. Since $V(x(t)) = J(\bar{x}^*(t),\cdot)$ and $V(x(t+\delta)) \leq J(\bar{x}^*(t+\delta,\bar{x}(t)),\cdot)$, the proposition follows.

Theorem 2: Suppose that Problem 1 is feasible at time t_0 with $\bar{x}(t_0) = x(t_0)$. Then, the system state x(t) converges to \mathcal{D}_H with probability p.

Proof: According to Eq. (15), $V(x(t_k))$ is monotonically non-increasing. Furthermore, the lower bound of $V(x(t_k))$ is 0. Thus, $V(x(t_k))$ converges as $t_k \to \infty$ for a bounded and monotonic function has a finite limit. By taking limits on both sides of Eq. (15),

$$\lim_{t_k \to \infty} \int_{t_k}^{t_k + \delta} \|\bar{x}^*(\tau, \bar{x}(t_k))\|_Q^2 d\tau$$

$$\leq \lim_{t_k \to \infty} V(x(t_k)) - \lim_{t_k \to \infty} V(x(t_k + \delta)) = 0$$

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Thus, $\bar{x}^*(\tau, \bar{x}(t_k)) \to 0$ for all $\tau \in [t_k, t_k + \delta]$ as $t_k \to \infty$. Since $x(t) - \bar{x}(t) \in \mathcal{D}_H$ with probability p for all $t \ge t_k$, $x(t) \to \mathcal{D}_H$ with probability p while $\bar{x}(t) \to 0$. \Box

The decreasing function V(x) converges to zero and the nominal system state \bar{x} approaches to its equilibrium. Thus, the real system state will approach to \mathcal{D}_H with probability p since $x(t) - \bar{x}(t) \in \mathcal{D}_H$.

V. ILLUSTRATIVE EXAMPLE

We consider a simple nonlinear system of the form

$$\dot{x}(t) = \begin{bmatrix} -1 & 2\\ -3 & 4 \end{bmatrix} x(t) + g(x(t)) + \begin{bmatrix} 0.5\\ -2 \end{bmatrix} u(t) + \begin{bmatrix} 0\\ 1 \end{bmatrix} w(t),$$
(16)

where $g(x) = \begin{bmatrix} 0 & -0.25x_2^3 \end{bmatrix}^T$. The origin of this system is open-loop unstable, while its linearized system is stabilizable. The system is with respect to chance constraints

$$P_r \{-2 \le u(t) \le 2\} \ge 0.97$$

$$P_r \{-3 \le x_i(t) \le 3\} \ge 0.97, \quad i = 1, 2.$$
(17)

The disturbance $w \in \mathbb{R}^1$ is assumed to be Gaussian random with E[w] = 0 and var[w] = 0.1.

A polytopic linear differential inclusion representation of the nonlinear system (16) is: $A_1 = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 2 \\ -3 & 3 \end{bmatrix}$, $B_1 = B_2 = \begin{bmatrix} 0.5 & -2 \end{bmatrix}'$, $B_{1w} = B_{2w} = \begin{bmatrix} 0 & 1 \end{bmatrix}'$. Solving (6) for the static case one obtains K and Γ , $K = \begin{bmatrix} -0.9226 & 3.6831 \end{bmatrix}'$ and $\Gamma = \begin{bmatrix} 0.1691 & 0.0619 \\ 0.0619 & 0.1279 \end{bmatrix}$.

A quadratic cost function $\tilde{l}(\bar{x}, \bar{u}) = \bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u}$ with Q = diag(0.5, 0.5) and R = 1 is used. Both the terminal control law and the terminal penalty matrix are computed via a convex optimization problem, c.f. [26], $\pi(\bar{x}) = \begin{bmatrix} -1.1365 & 1.1528 \end{bmatrix} \bar{x}$ and $S(\bar{x}) =$ $\bar{x}^T \begin{bmatrix} 8.8315 & -12.8757 \\ -12.8757 & 27.7728 \end{bmatrix} \bar{x}$. The resulting terminal set is $\mathcal{X}_f = \{\bar{x} \in \mathbb{R}^{n_2} \mid S(\bar{x}) \leq 10\}$. The open-loop optimization problem is solved at $\delta = 0.1$ sampling times using a prediction horizon of $T_p = 1.5$.

Figure 2 shows the state trajectory starting from the state $[-2.5 - 1.5]^T$ applying the proposed scheme. A total of 1000 simulations with var[w] = 0.1 were performed. As can be seen, the chance constraints (17) are satisfied with probability 1 although the "worst" (in the sense of the maximum amplitude) disturbance is |w| = 0.4289. This results from the fact that the controllability gramian is used to bound the covariance of v(t) for all $t \in [0, \infty)$, and the estimation on the covariance of v(t) is inevitably conservative.

For E[w] = 0 and var[w] = 0.1, $P_r\{w \in (-\infty, -0.3] \cup [0.3, \infty)\} = 0.0026$, i.e., "nearly all" values lie within a band around the mean a width of three standard deviations. Note that robust MPC schemes need to treat all possible disturbances no matter how small the probability of appearance is. In contract stochastic MPC approaches as shown only need to cope with "nearly all" disturbances which reduce the conservativeness in a probabilistic sense.

VI. CONCLUSIONS

This paper outlined a probabilistic model predictive control scheme for a class of continuous-time nonlinear systems with respect to stochastic disturbances and chance constraints. The control input is composed of both a nominal control action and an ancillary control law. The nominal control input is determined at each sampling time by a deterministic MPC formulation with tightened constraints. For the deterministic MPC formulation, Multivariate Chebyshev's inequality is used to reformulate the chance constraints to deterministic constraints. The nominal control action drives the mean of the original nonlinear systems to its equilibrium, while the ancillary control law maintains the error systems within a probabilistic set. The proposed scheme has almost the same computational burden as standard MPC for deterministic systems. Furthermore, asymptotic convergence in a probabilistic sense was proved. The price to pay is a slightly decrease in performance.

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Fig. 2. Exemplary time profiles for the system (16) with Algorithm 1 for disturbances w from $x_0 = [-2.5, -1.5]^T$, where E[w] = 0 and var[w] = 0.1.

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